FACTORIZATION OF PARAUNITARY POLYPHASE MATRICES USING SUBSPACE PROJECTIONS

Peter G. Vouras and Trac D. Tran

Naval Research Laboratory and Johns Hopkins University

ABSTRACT

Paraunitary filter banks (PUFBs) play an important role in multirate signal processing and image processing applications. In this paper a new factorization technique is presented based on the Singular Value Decomposition (SVD) that decomposes PUFBs into a product of elementary building blocks. These building blocks are parameterized by a set of angles that can be varied independently via optimization techniques to design a particular filter bank satisfying some criterion. The utility of this new matrix decomposition is that fewer free parameters are required to represent a PUFB as compared to conventional lattice factorizations, such as the Givens rotation matrix decomposition. The more economical PUFB representation presented in this paper improves the numerical behavior of nonlinear optimization programs used for designing PUFBs and allows for the design of longer channel filters without incurring additional computational complexity. A simulated design example is presented whereby a causal Finite Impulse Response (FIR) PUFB is designed to approximate an ideal, infinite order PUFB.

Index Terms— Paraunitary filter banks, decomposition, factorization, subspace projections, Singular Value Decomposition

1. INTRODUCTION

Consider an M-by-N polyphase transfer function matrix $\boldsymbol{H}(z)$. If $\boldsymbol{H}^{\mathbf{H}}(1/z^*)\boldsymbol{H}(z) = c^2\boldsymbol{I}$ for all values of z and c is some real nonzero constant, then the polyphase matrix $\boldsymbol{H}(z)$ is said to be paraunitary (PU). The paraunitary property implies that $\boldsymbol{H}(z)$ is unitary on the unit circle, i.e setting $z = e^{j\omega}$ yields $\boldsymbol{H}^{\mathbf{H}}(e^{j\omega})\boldsymbol{H}(e^{j\omega}) = c^2\boldsymbol{I}$, which implies that the system is lossless if each matrix entry $\boldsymbol{H}_{km}(z)$ is stable [1]. Paraunitary systems are desirable in a variety of signal and image processing applications. One of the many reasons is that the output energy of a paraunitary system is equal to the input energy, assuming that $c = 1$.

2. FACTORIZATION OF FIR PARAUNITARY MATRICES

In general, there are lattice factorizations and polynomial factorizations of PU polyphase matrices. A lattice factorization includes the standard Householder decomposition for an M-by-M PU matrix with degree N given by [2],

$$\boldsymbol{H}(z) = \boldsymbol{V_N}(z)\boldsymbol{V_{N-1}}(z)\cdots\boldsymbol{V_1}(z)\boldsymbol{U}$$

where $\boldsymbol{V}_{\mathrm{j}}(z) = \boldsymbol{I}_{\mathrm{M}} - \boldsymbol{v}_{\mathrm{j}}\boldsymbol{v}_{\mathrm{j}}^{\mathrm{H}} + z^{-1}\boldsymbol{v}_{\mathrm{j}}\boldsymbol{v}_{\mathrm{j}}^{\mathrm{H}}$ with $\boldsymbol{v}_{\mathrm{j}}$ a unit-norm vector and $\boldsymbol{U}$ a constant unitary matrix.

The other standard lattice decomposition of PU matrices is in terms of Givens rotation matrices as in,

$$\boldsymbol{H}(z) = \boldsymbol{R_N}\boldsymbol{\Lambda}(z)\boldsymbol{R_{N-1}}\boldsymbol{\Lambda}(z)\cdots\boldsymbol{\Lambda}(z)\boldsymbol{R_0}$$

where $\boldsymbol{R}_{\mathrm{j}}$ is a unitary product of $\frac{1}{2}M(M-1)$ Givens rotation matrices and $\boldsymbol{\Lambda}(z) = \mathrm{diag}(\boldsymbol{I}, z^{-1}\boldsymbol{I})$. It is also possible to perform a polynomial factorization of PU matrices using lifting steps [3]. For example, a 2-by-2 PU matrix may be decomposed into the product

$$\boldsymbol{H}(\boldsymbol{z}) = \prod_{k=1}^{N}\begin{bmatrix} 1 & s_k(z) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ t_k(z) & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 1/b \end{bmatrix}$$

where $s_{\mathrm{k}}(z)$, $t_{\mathrm{k}}(z)$ are Laurent polynomials and b is a non-zero constant.

3. FACTORIZATION BASED ON SUBSPACE PROJECTIONS

In this section, we propose a new factorization of PU matrices based on subspace projections computed using the Singular Value Decomposition (SVD). Reference [4] proves the existence of a factorization algorithm for FIR PU matrices based on projection operators, but it does not give an explicit construction as will be presented here.

First, consider a maximally decimated M-channel causal FIR PUFB with filter length $L = MK$. The M-by-M polyphase matrix of the analysis bank can be written as

$$\boldsymbol{H}(z) = \sum_{k=0}^{K-1}\boldsymbol{H_k}z^{-k}$$

where $\boldsymbol{H}_{\mathrm{k}}$ is a constant M-by-M matrix and $\boldsymbol{H}_{\mathrm{K-1}}$ is nonzero. Substituting this sum into the PU condition $\boldsymbol{H}^{\mathbf{H}}(1/z^*)\boldsymbol{H}(z) = \boldsymbol{H}(z)\boldsymbol{H}^{\mathbf{H}}(1/z^*) = c^2\boldsymbol{I}$ and equating like powers of z yields

$$\boldsymbol{H}_0^H \boldsymbol{H}_{K-1} = 0, \qquad (1)$$

$$\boldsymbol{H}_0 \boldsymbol{H}_{K-1}^H = 0 \qquad (2)$$

which implies that the matrices $\boldsymbol{H}_0$ and $\boldsymbol{H}_{K-1}$ are singular and that their ranks are less than M.

Consider the SVD of matrices $\boldsymbol{H}_0$ and $\boldsymbol{H}_{K-1}$ given by

$$\boldsymbol{H}_0 = \boldsymbol{U}_0 \boldsymbol{\Sigma}_0 \boldsymbol{V}_0^H, \quad \boldsymbol{H}_{K-1} = \boldsymbol{U}_{K-1} \boldsymbol{\Sigma}_{K-1} \boldsymbol{V}_{K-1}^H.$$

Let r denote the rank of $\boldsymbol{H}_0$. Then the orthogonal M-by-M matrices $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ can be partitioned as follows,

$$\boldsymbol{U}_0 = \left[\boldsymbol{U}_r^0 \middle| \tilde{\boldsymbol{U}}_r^0\right], \quad \boldsymbol{V}_0 = \left[\boldsymbol{V}_r^0 \middle| \tilde{\boldsymbol{V}}_r^0\right]$$

where $\boldsymbol{U}_r^0$ is size M-by-r, $\tilde{\boldsymbol{U}}_r^0$ is size M-by-(M-r), $\boldsymbol{V}_r^0$ is size M-by-r, and $\tilde{\boldsymbol{V}}_r^0$ is size M-by-(M-r). The matrix $\boldsymbol{\Sigma}_0 = \mathrm{diag}(\sigma_1,\ldots,\sigma_r,0,\ldots,0)$ contains the nonnegative singular values of $\boldsymbol{H}_0$. For a real matrix $\boldsymbol{H}_0$, the following remarks are valid [5]. The matrix $\boldsymbol{V}_r^0(\boldsymbol{V}_r^0)^T$ is the projection matrix onto $null(\boldsymbol{H}_0)^\perp = ran(\boldsymbol{H}_0^T) = span\{\boldsymbol{I} - \tilde{\boldsymbol{V}}_r^0(\tilde{\boldsymbol{V}}_r^0)^T\}$. The matrix $\tilde{\boldsymbol{V}}_r^0(\tilde{\boldsymbol{V}}_r^0)^T$ is the projection matrix onto $null(\boldsymbol{H}_0) = span\{\boldsymbol{I} - \boldsymbol{V}_r^0(\boldsymbol{V}_r^0)^T\}$. The matrix $\boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T$ is the projection matrix onto $ran(\boldsymbol{H}_0) = span\{\boldsymbol{I} - \tilde{\boldsymbol{U}}_r^0(\tilde{\boldsymbol{U}}_r^0)^T\}$. The matrix $\tilde{\boldsymbol{U}}_r^0(\tilde{\boldsymbol{U}}_r^0)^T$ is the projection matrix onto $null(\boldsymbol{H}_0^T) = ran(\boldsymbol{H}_0)^\perp = span\{\boldsymbol{I} - \boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T\}$. The same observations and notations are valid for $\boldsymbol{H}_{K-1}$ which has rank $q < M$.

The following facts will be useful in the sequel.

FACT 1. The columns of $\boldsymbol{U}_0$ are orthogonal to the columns of $\boldsymbol{U}_{K-1}$.

Proof. This fact is a direct consequence of equation (1).

FACT 2. The columns of $\boldsymbol{V}_0$ are orthogonal to the columns of $\boldsymbol{V}_{K-1}$.

Proof. This fact is a direct consequence of equation (2).

Now, define the order-one polynomial matrix in z^{-1},

$$\boldsymbol{P} = \boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T + \left[\boldsymbol{I} - \boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T\right] z^{-1}.$$

We have,

Proposition 1. The matrix $\boldsymbol{P}$ is paraunitary.

Proof. Using the fact that $(\boldsymbol{U}_r^0)^T \boldsymbol{U}_r^0 = \boldsymbol{I}$ and writing out the terms in the product $\boldsymbol{P}\boldsymbol{P}^H = \boldsymbol{P}^H\boldsymbol{P}$, the result is clear.

Furthermore,

Proposition 2. $\boldsymbol{P}$ is a left factor of $\boldsymbol{H}(z)$. In other words, $\boldsymbol{H}(z) = \boldsymbol{P}\boldsymbol{Q}(z)$ for some polynomial matrix $\boldsymbol{Q}(z)$ of reduced order K-2, where order denotes the highest power of z^{-1}.

Proof. Write the matrix product

$$\boldsymbol{P}^H \boldsymbol{H}(\boldsymbol{z}) = \left[\boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T + \left(\boldsymbol{I} - \boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T\right) z^{-1}\right]^H \times \left[\boldsymbol{H}_0 + \boldsymbol{H}_1 z^{-1} + \cdots + \boldsymbol{H}_{K-1} z^{-(K-1)}\right] \equiv \boldsymbol{Q}(\boldsymbol{z}).$$

Then $\boldsymbol{H}(z) = \boldsymbol{P}\boldsymbol{Q}(z)$. Using Fact 1 as well as the facts that $\boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T\boldsymbol{H}_0 = \boldsymbol{H}_0$, and $(\boldsymbol{I} - \boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T)\boldsymbol{H}_0 = 0$, and $(\boldsymbol{I} - \boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T)\boldsymbol{H}_{K-1} = \boldsymbol{H}_{K-1}$, one finds that $\boldsymbol{Q}(z)$ is of reduced order K-2.

Proposition 3. The matrix $\boldsymbol{Q}(z)$ is paraunitary.

Proof. It is straightforward to verify that $\boldsymbol{Q}(z)\boldsymbol{Q}(z)^H = \boldsymbol{Q}(z)^H\boldsymbol{Q}(z) = \boldsymbol{I}$.

Propositions 1, 2, and 3 suggest that the polyphase matrix $\boldsymbol{H}(z)$ can be decomposed into a product of elementary paraunitary matrices $\boldsymbol{P}_k$ by iteratively applying the procedure above to extract left factors until the remaining matrix is a paraunitary matrix of order one. Conversely, the matrix product

$$\boldsymbol{H}(z) = \boldsymbol{P}_L \boldsymbol{P}_{L-1} \cdots \boldsymbol{P}_1 \boldsymbol{T}$$

with $\boldsymbol{T}$ an arbitrary constant unitary matrix, is a causal FIR paraunitary system.

The McMillan degree (often just called degree) of the filter bank structure described by $\boldsymbol{P}$ is the smallest number of delays with which the system can be implemented. If $\boldsymbol{H}(z) = \boldsymbol{H}_0 + z^{-1}\boldsymbol{H}_1$ with $\boldsymbol{H}_1 \neq \boldsymbol{0}$, then its order equals one, whereas its degree is equal to the rank of $\boldsymbol{H}_1$ [6]. Thus the degree of $\boldsymbol{P}$ is M-r and $\boldsymbol{P}$ is minimal if and only if it is of the form,

$$\boldsymbol{P} = \boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T + \left[\boldsymbol{I} - \boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T\right] z^{-(M-r)}.$$

An alternative formulation of the decomposition can be derived by using the matrix,

$$\boldsymbol{P} = \tfrac{1}{\sqrt{2}}\left(\boldsymbol{U}_r^0 \left(\boldsymbol{U}_r^0\right)^T + \tilde{\boldsymbol{U}}_r^0 \left(\tilde{\boldsymbol{U}}_r^0\right)^T z^{-1}\right).$$

This version of $\boldsymbol{P}$ is also paraunitary, with or without the scale factor, and is also a left factor of $\boldsymbol{H}(z)$ such that $\boldsymbol{H}(z)$ can be written as $\boldsymbol{H}(z) = \boldsymbol{P}\boldsymbol{Q}(z)$ where $\boldsymbol{Q}(z)$ is of reduced order and paraunitary.

4. PARAMETERIZATION

In order to design practical filter banks, it is necessary to parameterize the building blocks $\boldsymbol{P}$ in terms of variables which can be tuned to optimize an objective function, such as the Mean Square Error (MSE) between the desired filter bank and an approximation.

Since $\boldsymbol{U}_r^0$ is orthonormal, it can be written using the Cosine-Sine decomposition as [5],

$$\boldsymbol{U}_r^0 = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{S} \end{bmatrix} \boldsymbol{V}_1^T,$$

where $\boldsymbol{U}_1$ is M_1-by-M_1, $\boldsymbol{U}_2$ is M_2-by-M_2, $\boldsymbol{C}$ is M_1-by-r, $\boldsymbol{S}$ is M_2-by-r, and $\boldsymbol{V}_1$ is r-by-r. Furthermore, $M_1 \geq r$, $M_2 \geq r$, and $M_1 + M_2 = M$. The matrices $\boldsymbol{C}$ and $\boldsymbol{S}$ satisfy the condition that $\boldsymbol{C}^T\boldsymbol{C} + \boldsymbol{S}^T\boldsymbol{S} = \boldsymbol{I}$ and can be taken to be $\boldsymbol{C} = \mathrm{diag}(\cos(\theta_1), \ldots,\cos(\theta_r),0, \ldots,0)$ and $\boldsymbol{S} = \mathrm{diag}(\sin(\theta_1), \ldots, \sin(\theta_r),0, \ldots, 0)$. The matrices $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and $\boldsymbol{V}_1$ are orthogonal and can be parameterized in terms of Givens rotations. Using this decomposition, a paraunitary building block of order one with $M = 4$ channels can be parameterized in terms of 5 angles, as opposed to 6 angles if the standard decomposition in terms of Givens rotation matrices is used.

An alternative implementation of $\boldsymbol{P}$ can be derived by defining the unitary matrix

$$\boldsymbol{X} = \left[\boldsymbol{U}_r^0 \middle| \tilde{\boldsymbol{U}}_r^0 \right].$$

Then,

$$\boldsymbol{P} = \boldsymbol{X}\boldsymbol{\Lambda}(z)\boldsymbol{X}^T, \quad \boldsymbol{\Lambda}(z) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & z^{-1}\boldsymbol{I} \end{bmatrix}.$$

Now any PU filter bank can be written as the product

$$\boldsymbol{H}(z) = \boldsymbol{X}_L\boldsymbol{\Lambda}(z)\boldsymbol{X}_L^T\boldsymbol{X}_{L-1}\boldsymbol{\Lambda}(z)\boldsymbol{X}_{L-1}^T\cdots\boldsymbol{X}_1\boldsymbol{\Lambda}(z)\boldsymbol{X}_1^T$$

$$= \boldsymbol{U}_L\boldsymbol{\Lambda}(z)\boldsymbol{U}_{L-1}\boldsymbol{\Lambda}(z)\cdots\boldsymbol{U}_1\boldsymbol{\Lambda}(z)\boldsymbol{U}_0,$$

$$\boldsymbol{U}_j = \begin{Bmatrix} \boldsymbol{X}_j & j = L \\ \boldsymbol{X}_1^T & j = 0 \\ \boldsymbol{X}_{j+1}^T\boldsymbol{X}_j & \text{otherwise} \end{Bmatrix}.$$

The matrices $\boldsymbol{U}_j$ are unitary because the matrices $\boldsymbol{X}_j$ are unitary. This form of the factorization is similar to a form presented in [2] and derived differently.

5. RESULTS

The foremost advantage of the new polyphase matrix factorization presented in this paper is that it is a more economical representation than the standard Givens decomposition. The standard Givens polyphase matrix decomposition is given by,

$$\boldsymbol{H}(z) = \boldsymbol{G}_L\boldsymbol{\Lambda}(z)\cdots\boldsymbol{G}_1\boldsymbol{\Lambda}(z)\boldsymbol{Q}\boldsymbol{J}$$

where L is the McMillan degree of $\boldsymbol{H}(z)$, $\boldsymbol{G}_k$ and $\boldsymbol{Q}$ are each the product of $\frac{1}{2}M(M-1)$ Givens rotation matrices, $\boldsymbol{\Lambda}(z) = \text{diag}(\boldsymbol{I}, z^{-1}\boldsymbol{I})$ and $\boldsymbol{J} = \text{diag}(+/-1, \ldots, +/-1)$. The total number of free parameters, or rotation angles, necessary to represent a polyphase matrix of McMillan degree L using the Givens decomposition is $\frac{1}{2}(L+1)M(M-1)$. The channel filters in the Givens filter bank representation will have a length equal to ML.

Using the proposed new factorization procedure and a Givens decomposition, Fig. 1 compares the length of the channel filters and the number of angles used to parameterize the synthesis polyphase matrix for a 4-channel system. As is evident from the figure, the proposed new polyphase matrix decomposition is much more scalable and allows for the design of longer filters using fewer free parameters. This representational efficiency is important because it allows for the design of longer channel filters without increasing the computational complexity or degrading the numerical behavior of the nonlinear optimization program used in designing the filter bank. Note that valid filter lengths in Fig. 1 are multiples of $M = 4$.

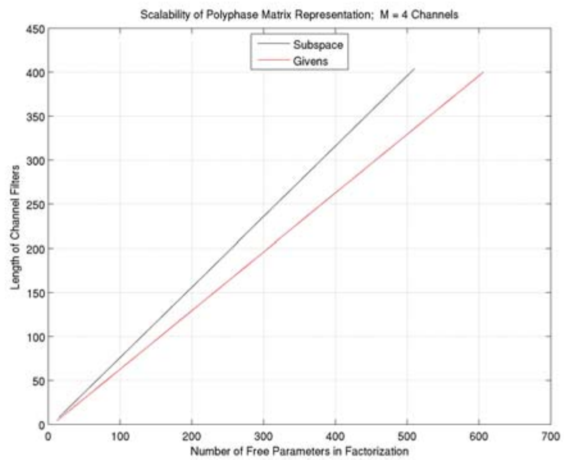

Fig. 1. Scalability of Polyphase Matrix Representation

The parameterized form of the polyphase matrix decomposition based on subspace projections was used to design a causal FIR PU approximation to an ideal Principal Component Filter Bank (PCFB). PCFBs are described in [7] and [8]. An ideal PCFB corresponds to an infinite order PU filter bank. This ideal filter bank can be approximated using a FIR PU filter bank by minimizing the weighted mean squared Frobenius norm error between the desired polyphase matrix of the ideal PCFB, $\boldsymbol{D}(\omega)$, and the FIR PU synthesis polyphase matrix, $\boldsymbol{H}(\omega)$, as in

$$\min\ \eta = \frac{1}{2\pi}\int_0^{2\pi} W(\omega)\left\|\boldsymbol{D}(\omega) - \boldsymbol{H}(\omega)\right\|_F^2 d\omega$$

where $W(\omega)$ is a scalar nonnegative weight function. Figures 2 and 3 illustrate two of the channel filters achieved for a system with $M = 4$ channels. Each channel filter has 24 taps. The rank of the matrix $\boldsymbol{U}_r^0(\boldsymbol{U}_r^0)^T$ was two (i.e. $r = 2$). The brick-wall, ideal filter responses are designated using dotted lines. Included in the figures for comparison and illustrated using black lines, are the filters designed using the greedy algorithm described by Tkacenko in [8]. As the figures show, the polyphase matrix decomposition based on subspace projections yields channel filters that compare favorably with those designed using algorithms recently reported in the literature.

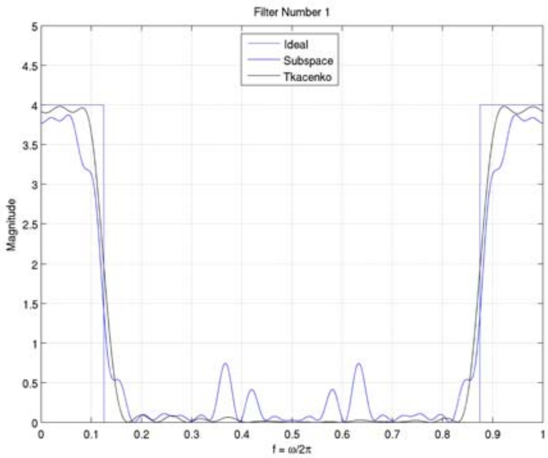

Fig. 2. Filter 1

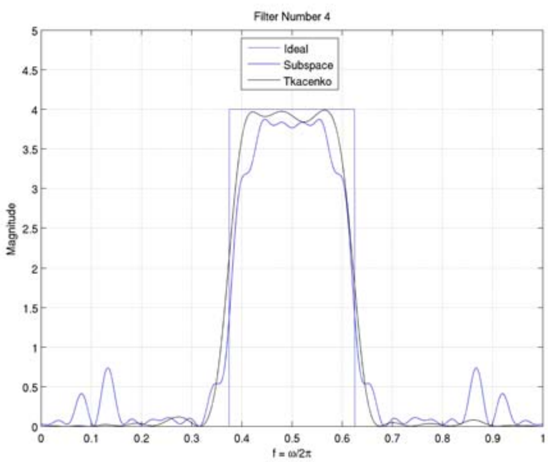

Fig. 3. Filter 4

6. CONCLUSIONS

A new decomposition procedure for FIR PUFBs based on the use of projection matrices was presented in this paper. This factorization of a FIR PU polyphase matrix into a product of elementary PU building blocks can be parameterized in terms of fewer rotation angles than the standard Givens factorization. The rotation angles can then be varied as design parameters in an optimization routine to yield a practical system that satisfies a given design criterion. Sample simulation results were presented which demonstrate the excellent passband performance of channel filters designed using this method.

7. REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, NJ, 1993.

[2] X. Gao, T. Nguyen, "On Factorization of M-Channel Paraunitary Filterbanks," *IEEE Transactions on Signal Processing*, pp. 1433-1446, July 2001.

[3] I. Daubechies and W. Sweldens, "Factoring Wavelet Transforms Into Lifting Steps," manuscript, November 1997.

[4] S. Lannes and N. K. Bose, "A Factorization Procedure for FIR Paraunitary Matrices," *Proceedings International Conference on Acoustics, Speech, and Signal Processing*, Philadelphia, PA, 2005.

[5] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, 1996.

[6] S. Phoong and P. P. Vaidyanathan, "Paraunitary Filter Banks Over Finite Fields," *IEEE Transactions on Signal Processing*, pp. 1443-1457, June 1997.

[7] M. K. Tsatsanis and G. B. Giannakis, "Principal Component Filter Banks for Optimal Multiresolution Analysis," *IEEE Transactions on Signal Processing*, pp. 1766-1777, Aug. 1995.

[8] A. Tkacenko and P. P. Vaidyanathan, "Iterative Greedy Algorithm for Solving the FIR Paraunitary Approximation Problem," *IEEE Transactions on Signal Processing*, pp. 146-160, Jan 2006.