Signal Adapted Filter Bank Design Using Markov Parameters

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SUMMARY

In this paper a method for designing Principal Component Filter Banks (PCFBs) based on a truncated sequence of estimated Markov parameters is described. This technique yields filters that are an excellent approximation to the ideal PCFB channel filters without nearly the computational complexity of traditional numerical optimization methods.

1. *BACKGROUND*

PCFBs were originally described in [1]. Consider the blocked version of the input sequence to a filter bank to be the *M*-by-1 vector

$$\boldsymbol{x}(n) = \begin{bmatrix} x(nM) & x(nM-1) & \cdots & x(nM-M+1) \end{bmatrix}^T$$

and define the blocked output vector to be

$$\hat{\boldsymbol{x}}(n) = \begin{bmatrix} \hat{x}(nM) & \hat{x}(nM-1) & \cdots & \hat{x}(nM-M+1) \end{bmatrix}^T.$$

A PCFB is the solution to the problem of finding optimal *Q*-by-*M* and *M*-by-*Q* analysis and synthesis polyphase matrices, $\boldsymbol{E}(n)$ and $\boldsymbol{R}(n)$ with $Q < M$, such that the time-averaged mean squared error between the vector input and output is minimized, as in

$$\min \xi = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} E\left\{ \left[\boldsymbol{x}(n) - \hat{\boldsymbol{x}}(n)\right]^T \left[\boldsymbol{x}(n) - \hat{\boldsymbol{x}}(n)\right] \right\}.$$

The solution polyphase matrices are given by

$$\boldsymbol{E}(\omega) = \begin{bmatrix} \boldsymbol{v}_1^H(\omega) \\ \vdots \\ \boldsymbol{v}_Q^H(\omega) \end{bmatrix}$$

$$\boldsymbol{R}(\omega) = \boldsymbol{E}^H(\omega)$$

where $\boldsymbol{v}_i(\omega)$ is the eigenvector corresponding to the *i*th largest eigenvalue of $\hat{\boldsymbol{S}}_{xx}(\omega)$. $\hat{\boldsymbol{S}}_{xx}(\omega)$ is the time averaged spectral density matrix of the input vector process. Using this construction for the polyphase matrices $\boldsymbol{E}(\omega)$ and $\boldsymbol{R}(\omega)$ when $Q = M$ results in a paraunitary (PU) *M*-channel perfect reconstruction filter bank, albeit one of infinite order.

PCFBs have special properties but they only exist for special classes of filter banks [2]. If a PCFB exists for an input power spectral density (PSD) matrix $\boldsymbol{S}_{xx}(\omega)$ and for a class *C* of filter banks, then the subband variance vector of the PCFB,

$$\boldsymbol{\sigma} = \left[\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2\right]^T$$

majorizes the subband variance vector created by any other filter bank in *C*. If *C* is the class of all paraunitary filter banks with infinite order, then the PCFB exists and is the pointwise in frequency Karhunen-Loève transform (KLT) for $\boldsymbol{x}(n)$. In other words, $\boldsymbol{R}(\omega)$ diagonalizes and thereby totally decorrelates $\boldsymbol{S}_{xx}(\omega)$ for every ω such that the frequency dependent eigenvalues are always arranged in decreasing order.

It has been shown that PCFBs are optimal for maximizing coding gain and minimizing mean-squared error in the presence of quantization noise, but they are also optimal for any concave objective function of the subband variance vector [3].

2. *MARKOV PARAMETER SEQUENCES*

The channel filters of ideal PCFBs have a brick-wall response which can only be approximated using finite impulse response (FIR) filter banks. In this paper, we will describe a method for designing FIR approximations to PCFBs using a truncated sequence of Markov parameters.

Any transfer function or polyphase matrix representation $\boldsymbol{G}(z)$ can be written in terms of the z-transform of the unit pulse response,

$$\mathbf{G}(z) = \mathbf{G_0} + \mathbf{G_1}z^{-1} + \mathbf{G_2}z^{-2} + \mathbf{G_3}z^{-3} + \ldots$$

The matrices $\boldsymbol{G}(\text{n})$ are known as Markov parameters. For a causal system, the Markov parameter sequence is equivalent to the impulse response. In general, the Markov parameter sequence is infinite.

Given $\boldsymbol{G}(z)$, the filter bank design method proposed in this paper consists of truncating the Markov parameter sequence of the ideal or desired transfer function to a finite length *N*, as in

$$\mathbf{G}_T(z) = \mathbf{G_0} + \mathbf{G_1}z^{-1} + \ldots + \mathbf{G}_{N-1}z^{-(N-1)}.$$

Truncating the infinite Markov parameter sequence is an optimal method for designing causal filter banks in that it minimizes the square Frobenius norm error between the desired ideal filter bank and the approximation.

<u>Claim</u>: Given *N*, the filter bank obtained by Markov parameter sequence truncation has the least square error among all causal filter banks of order *N*-1 and length *N*.

<u>Proof</u>: Let $\boldsymbol{G}(\text{n})$, $0 \le n \le N-1$, be the Markov parameter sequence of length *N* for a causal transfer matrix of order *N*-1. The square error, ε^2, between the ideal infinite order filter bank $\boldsymbol{G}_{\mathbf{d}}(\text{n})$ and $\boldsymbol{G}(\text{n})$ can be written as,

$$\varepsilon^2 = \sum_{n=0}^{\infty} \left\| \mathbf{G_d}(n) - \mathbf{G}(n) \right\|_F^2$$

$$= \sum_{n=0}^{N-1} \left\| \mathbf{G_d}(n) - \mathbf{G}(n) \right\|_F^2 + \sum_{n=N}^{\infty} \left\| \mathbf{G_d}(n) \right\|_F^2 .$$

The second term is not affected by the choice of $\boldsymbol{G}$(n). Thus, the square error is minimized if and only if the nonnegative first term equals zero. This implies that $\boldsymbol{G_d}$(n) = $\boldsymbol{G}$(n) for $0 \le n \le N\text{-}1$, which is precisely the filter design criteria.

3. *FILTER BANK DESIGN METHOD*

Given the ideal polyphase matrix $\boldsymbol{G}(z)$ sampled at $z = e^{j2\pi k/N}$, for $0 \le k \le N\text{-}1$, define

$$\mathbf{G}(k) = \mathbf{G}(z)\Big|_{z=e^{\frac{j2\pi k}{N}}}$$

$$= \mathbf{G_0} + e^{-\frac{j2\pi k}{N}}\mathbf{G_1} + e^{-\frac{j4\pi k}{N}}\mathbf{G}_2 + \cdots + e^{-\frac{j2\pi k(N-1)}{N}}\mathbf{G}_{N-1} .$$

The Markov parameters $\boldsymbol{G}$(n) can be recovered as

$$\mathbf{G}(n) = \frac{1}{N}\sum_{k=0}^{N-1}\mathbf{G}(k)e^{\frac{j2\pi kn}{N}},$$

for $0 \le n \le N\text{-}1$. Next, by appending zeros to the end of the sequence $\boldsymbol{G}$(n) as necessary, find three positive integers l, q, and n such that the following rank condition is satisfied,

$$\textbf{rank}\ \Gamma_{lq} = \textbf{rank}\ \Gamma_{l+1,\,q+j} = n, \qquad j = 1, 2, \ldots$$

The matrix $\boldsymbol{\Gamma}$ is the block Hankel matrix defined as

$$\Gamma_{lq} = \begin{bmatrix} \mathbf{G_1} & \mathbf{G_2} & \cdots & \mathbf{G_q} \\ \mathbf{G_2} & \mathbf{G_3} & \cdots & \mathbf{G_{q+1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G_l} & \mathbf{G_{l+1}} & \cdots & \mathbf{G_{l+q-1}} \end{bmatrix}.$$

The following steps will find matrices ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) corresponding to the linear state equation

$$\mathbf{x}(n+1) = \mathbf{Ax}(n) + \mathbf{Bu}(n)$$

$$\mathbf{y}(n) = \mathbf{Cx}(n) .$$

The transfer function matrix of this linear system is

$$\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} ,$$

which will be a close approximation to the ideal polyphase matrix $\boldsymbol{G}(z)$. If the number of channels in the filter bank is m, then the matrix $\boldsymbol{A}$ is size n-by-n, $\boldsymbol{B}$ is n-by-m, and $\boldsymbol{C}$ is m-by-n. Once ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) are computed, the coefficients of the filter bank channel filters can be recovered from the MIMO impulse response $\boldsymbol{H}(n)$ using

$$\mathbf{H}(n) = \mathbf{CA}^{n-1}\mathbf{B}, \qquad n = 1, 2, \ldots, N .$$

In general, filters designed using this technique will have complex coefficients. Useful background on linear system theory can be found in [4] and [5].

To find ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) let

a) $\boldsymbol{H_q}$ denote the n-by-qm submatrix formed from the first n linearly independent rows of $\boldsymbol{\Gamma_{l,q}}$.

b) Let $\boldsymbol{H_q^s}$ be another n-by-qm submatrix. The ith row of $\boldsymbol{H_q^s}$ is the row of $\boldsymbol{\Gamma_{l+1,q}}$ that is m rows below the row of $\boldsymbol{\Gamma_{l+1,q}}$ that is the ith row of $\boldsymbol{H_q}$.

c) Let $\boldsymbol{F}$ be the invertible n-by-n matrix comprising the first n linearly independent columns of $\boldsymbol{H_q}$.

d) Let $\boldsymbol{F_s}$ be the n-by-n matrix occupying the same column positions in $\boldsymbol{H_q^s}$ as does $\boldsymbol{F}$ in $\boldsymbol{H_q}$.

e) Let $\boldsymbol{F_c}$ be the m-by-n matrix occupying the same column positions in $\boldsymbol{\Gamma_{1,q}}$ as does $\boldsymbol{F}$ in $\boldsymbol{H_q}$.

f) Let $\boldsymbol{F_r}$ be the n-by-m matrix comprising the first m columns of $\boldsymbol{H_q}$.

Now define

$$\mathbf{A} = \mathbf{F_s F^{-1}}$$

$$\mathbf{B} = \mathbf{F_r}$$

$$\mathbf{C} = \mathbf{F_c F^{-1}}.$$

4. *SIMULATION RESULTS*

The performance of this design method was demonstrated using an AR(4) signal with the PSD illustrated in Fig. 1.

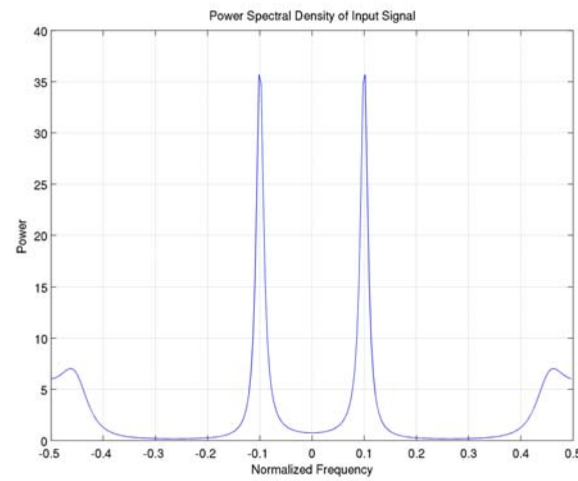

Fig. 1. Input Signal PSD

Frequency domain samples of the ideal synthesis polyphase matrix, $\boldsymbol{G}(k)$, were calculated by computing the eigenvector decomposition of the PSD matrix $\boldsymbol{S_{xx}}(\omega)$ for a discrete set of N frequencies, and arranging the eigenvectors into the columns of $\boldsymbol{G}(k)$ such that they correspond to a decreasing ordering of eigenvalues. For this example, N was chosen to be 64. The scalar n that satisfied the rank condition was 254. The number of channels, m, in the filter bank is 4 and the length of each channel filter is 256 complex coefficients. Note that since the dimensionality of the state vector $\boldsymbol{x}(n)$ is 254 and the McMillan degree of $\boldsymbol{H}(z)$ is 4, the realization of $\boldsymbol{H}(z)$ given by ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$) is not minimal. Figures 2 through 5 illustrate the frequency response of each computed channel filter. Superimposed on each plot is the brick-wall response of the ideal filter. As the plots clearly demonstrate, all the channel filters are excellent approximations to the ideal filters over the discrete set of N sample frequencies.

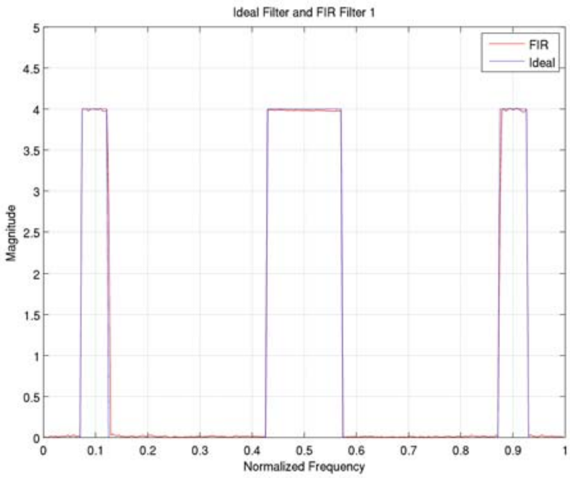

Fig. 2. Filter 1

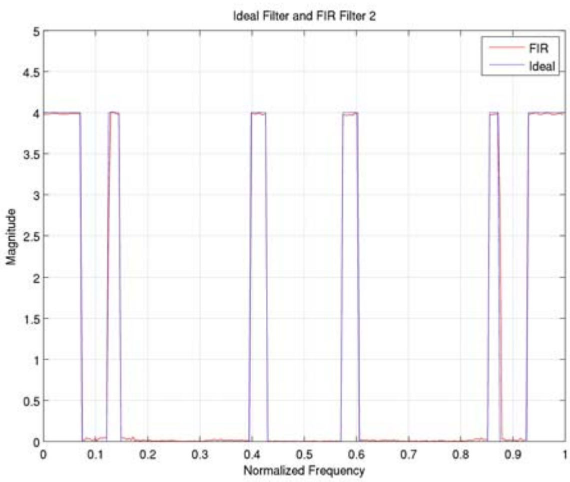

Fig. 3. Filter 2

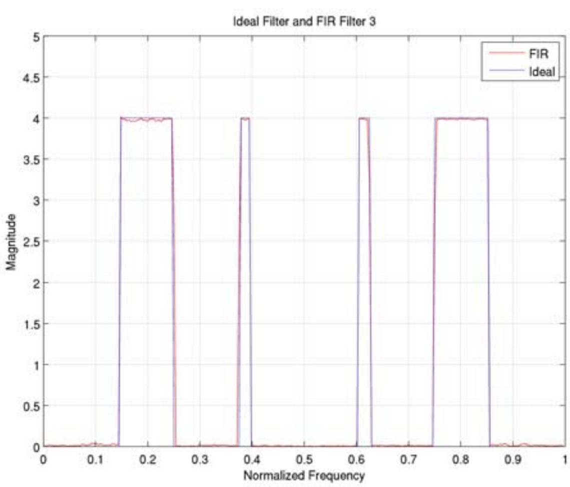

Fig. 4. Filter 3

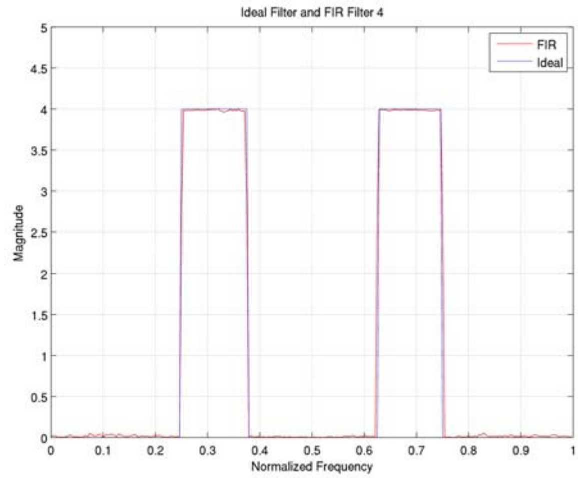

Fig. 5. Filter 4

5. *CONCLUSIONS*

In this paper, a new technique for approximating ideal PCFBs adapted to the characteristics of an input signal is described. The algorithm is based on calculating a finite-length sequence of Markov parameters and yields causal FIR channel filters of any length with very small errors compared to the ideal frequency response. For designing filter banks with long channel filters that must encompass multiple highly selective pass-bands, this algorithm is far superior to conventional numerical optimization techniques because of its computational simplicity.

6. *REFERENCES*

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