Design of Pointwise-in-Frequency Paraunitary Filter Banks

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SUMMARY

Designing paraunitary filter banks (PUFBs) that minimize a mean squared error (MMSE) design criterion and are globally optimal is a difficult problem in nonlinear optimization. Many techniques have been proposed to simplify this problem including the use of reduced parameter PUFB decompositions to reduce the complexity of the nonlinear program. In this paper, a simple yet effective method is presented for designing PUFBs that satisfy an approximated MMSE criterion with the caveat that the paraunitary condition is satisfied only on a discrete set of frequencies. No statement can be made about the performance or the losslessness of the designed filter bank between selected frequency points, but for a sufficiently large grid with a dense distribution of points, the proposed design method may yield adequate performance in practical situations.

1. *PARAUNITARY FILTER BANKS*

A p-by-r rational transfer function $\mathbf{H}(z)$, with $p \geq r$, satisfies the paraunitary condition if

$$\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = c^2\mathbf{I}, \quad \forall\, z,$$

where c is a real, nonzero constant, the complex variable $z = re^{j\omega}$ and $\widetilde{\mathbf{H}}(z) = \mathbf{H}^H\left(1/z^*\right)$. The paraunitary condition implies the lossless property. Namely, a transfer matrix $\mathbf{H}(z)$ corresponding to a causal system is called lossless if each matrix entry $[\mathbf{H}(z)]_{ij}$ is stable and $\mathbf{H}(z)$ is unitary on the unit circle, meaning that with $z = e^{j\omega}$,

$$\mathbf{H}^H\left(e^{j\omega}\right)\mathbf{H}\left(e^{j\omega}\right) = c^2\mathbf{I}, \quad \forall\, \omega .$$

For rational transfer functions, the lossless property implies the paraunitary condition. Therefore, a lossless system can be defined to be a causal, stable paraunitary system [1]. Paraunitary systems are highly desirable in a variety of signal processing applications precisely because of their lossless property.

2. *PARAMETERIZED DECOMPOSITIONS OF PARAUNITARY FILTER BANKS*

Causal finite impulse response (FIR) PUFBs may be decomposed into a product of elementary building blocks [1]. Two such decompositions are considered standard in the literature. The first decomposition of $\mathbf{H}(z)$ is based on Householder-like building blocks as in,

$$\mathbf{H}(z) = \left[\prod_{j=1}^{N} \mathbf{V_j}(z)\right]\mathbf{Q}$$

where N is the Smith-McMillan degree of $\mathbf{H}(z)$, $\mathbf{Q}$ is a p-by-r unitary matrix and the p-by-p matrix

$$\mathbf{V_j}(z) = \mathbf{I} - \mathbf{v_j}\mathbf{v_j}^H + z^{-1}\mathbf{v_j}\mathbf{v_j}^H$$

with $\mathbf{v}_j$ a unit norm vector.

The second common factorization of $\mathbf{H}(z)$ is in terms of Givens-like building blocks. For instance,

$$\mathbf{H}(z) = \mathbf{R}_N\mathbf{\Lambda}(z)\cdots\mathbf{R}_1\mathbf{\Lambda}(z)\mathbf{Q}$$

where N is the Smith-McMillan degree of $\mathbf{H}(z)$, $\mathbf{Q}$ is a p-by-r unitary matrix, $\mathbf{R}_j$ is the p-by-p product of ½p(p - 1) Givens rotation matrices and $\mathbf{\Lambda}(z)$ = diag($\mathbf{I}$, $z^{-1}\mathbf{I}$). Each Givens rotation matrix is of the form

$$\mathbf{R}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta_j) & \cdots & \sin(\theta_j) & 0 \\ 0 & \vdots & 1 & \vdots & 0 \\ 0 & -\sin(\theta_j) & \cdots & \cos(\theta_j) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By using a Householder decomposition, $\mathbf{H}(z)$ can be parameterized in terms of the components of each vector $\mathbf{v}_j$. With a Givens decomposition, $\mathbf{H}(z)$ can be parameterized in terms of rotation angles.

3. *MINIMUM MEAN SQUARED ERROR DESIGN*

Given a desired p-by-r transfer function matrix, $\mathbf{D}(\omega)$, with response specified in the frequency domain, a p-by-r causal FIR approximation $\mathbf{H}(\omega)$ may be derived by minimizing the weighted mean-squared Frobenius norm error between $\mathbf{D}(\omega)$ and $\mathbf{H}(\omega)$ given by

$$\varepsilon = \tfrac{1}{2\pi}\int_0^{2\pi} W(\omega)\left\|\mathbf{D}(\omega) - \mathbf{H}(\omega)\right\|_F^2 d\omega ,$$

with $W(\omega)$ a scalar nonnegative weight function. Recall, the Frobenius norm squared of a matrix is equivalent to $\|\mathbf{H}\|_F^2 = tr\left[\mathbf{H}^H\mathbf{H}\right]$. By embedding a parameterized Householder or Givens factorization for $\mathbf{H}(\omega)$ into the integral, the MSE may be minimized using popular unconstrained minimization techniques, such as the Broyden, Fletcher, Goldfarb, and Shanno (BFGS) algorithm. The drawback to this approach is that the objective function is not convex and it is unlikely the algorithm will converge to a global solution. Furthermore, for large values of N, p, or r, the number of design parameters, be it components of $\mathbf{v}_j$ or rotation angles θ_j depending on the PU decomposition chosen, becomes overwhelmingly large and the computational complexity difficult to overcome.

4. *MARKOV PARAMETER SEQUENCES*

Any transfer function or polyphase matrix representation $\mathbf{H}(z)$ may be written in terms of the z-transform of the unit pulse response as in [2],

$$\mathbf{H}(z) = \mathbf{H_0} + \mathbf{H_1}z^{-1} + \mathbf{H_2}z^{-2} + \mathbf{H}_3 z^{-3} + \ldots$$

The p-by-r matrices $\mathbf{H}_n$ are known as Markov parameters. For a causal system, the Markov parameter sequence is equivalent to the impulse response. In general, the Markov parameter sequence is infinite.

The Markov parameter sequence corresponds to a linear time invariant state equation

$$\mathbf{x}(n+1) = \mathbf{Ax}(n) + \mathbf{Bu}(n)$$
$$\mathbf{y}(n) = \mathbf{Cx}(n) + \mathbf{Du}(n).$$

If $\mathbf{D} = \mathbf{0}$, then $\mathbf{H}_0 = \mathbf{0}$. If $\mathbf{H}(z)$ is p-by-r and $\mathbf{x}(n)$ is m-by-1, then the matrix $\boldsymbol{A}$ is size m-by-m, $\boldsymbol{B}$ is m-by-r, $\boldsymbol{C}$ is p-by-m and $\mathbf{D}$ is p-by-r [2].

Assuming $\mathbf{D} = \mathbf{0}$, the transfer function matrix of the linear system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is

$$\mathbf{H}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\ .$$

If the matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are given, the coefficients of the filter bank channel filters can be recovered from the MIMO impulse response $\mathbf{G}(n)$ using

$$\mathbf{G}(n) = \mathbf{CA}^{n-1}\mathbf{B}, \quad n = 1, 2, \ldots.$$

In general, these filters will have complex coefficients. A linear time invariant state equation with matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a realization of a Markov parameter sequence $\mathbf{H}_n$ if and only if

$$\mathbf{H}_n = \mathbf{CA}^{n-1}\mathbf{B}, \quad n = 1, 2, \ldots.$$

Using Markov parameters, the block Hankel matrix $\mathbf{\Gamma}_{lq}$ is defined as

$$\mathbf{\Gamma}_{lq} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_q \\ \mathbf{H}_2 & \mathbf{H}_3 & \cdots & \mathbf{H}_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_l & \mathbf{H}_{l+1} & \cdots & \mathbf{H}_{l+q-1} \end{bmatrix}.$$

The positive integers l and q are chosen such that the following rank condition is satisfied for a positive integer n, with $l, q \le n$,

$$\text{rank}(\mathbf{\Gamma}_{lq}) = \text{rank}(\mathbf{\Gamma}_{l+1,q+j}) = n, \quad j = 1, 2, \ldots.$$

Given $\mathbf{\Gamma}_{lq}$, to find the matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ of a minimal realization of dimension n, (i.e a realization which is both reachable and observable), let [2]

a) $\mathbf{H}_q$ denote the n-by-qr submatrix formed from the first n linearly independent rows of $\mathbf{\Gamma}_{lq}$.
b) Let $\mathbf{H}_q^s$ be another n-by-qr submatrix. The ith row of $\mathbf{H}_q^s$ is the row of $\mathbf{\Gamma}_{l+1,q}$ that is p rows below the row of $\mathbf{\Gamma}_{l+1,q}$ that is the ith row of $\mathbf{H}_q$.
c) Let $\mathbf{F}$ be the invertible n-by-n matrix comprising the first n linearly independent columns of $\mathbf{H}_q$.
d) Let $\mathbf{F}_s$ be the n-by-n matrix occupying the same column positions in $\mathbf{H}_q^s$ as does $\mathbf{F}$ in $\mathbf{H}_q$.
e) Let $\mathbf{F}_c$ be the p-by-n matrix occupying the same column positions in $\mathbf{\Gamma}_{1,q}$ as does $\mathbf{F}$ in $\mathbf{H}_q$.
f) Let $\mathbf{F}_r$ be the n-by-r matrix comprising the first r columns of $\mathbf{H}_q$.

Now define

$$\mathbf{A} = \mathbf{F}_s\mathbf{F}^{-1}$$
$$\mathbf{B} = \mathbf{F}_r$$
$$\mathbf{C} = \mathbf{F}_c\mathbf{F}^{-1}.$$

5. *DERIVATION OF POINTWISE-IN-FREQUENCY PARAUNITARY FILTER BANK*

Suppose we approximate the MSE ε using the Riemann sum as,

$$\hat{\varepsilon} = \sum_{k=0}^{N-1} \|\mathbf{D}(\omega_k) - \mathbf{H}(\omega_k)\|_F^2 \Delta\omega$$

where $\omega_k = 2\pi k/N$ and $\Delta\omega = 2\pi/N$. For large values of N, $\hat{\varepsilon}$ will be a close approximation to ε. Since each term in the summation is nonnegative, $\hat{\varepsilon}$ can be minimized by minimizing each term independently. To simplify notation, let $\mathbf{D}_k$ denote the p-by-r matrix $\mathbf{D}(\omega_k)$ and let $\mathbf{H}_k$ denote the p-by-r matrix $\mathbf{H}(\omega_k)$, $p \ge r$. Then the objective becomes to minimize

$$\hat{\varepsilon}_k = \|\mathbf{D}_k - \mathbf{H}_k\|_F^2$$

for $k = 0, \ldots, N$-1 and such that $\mathbf{H}_k^H\mathbf{H}_k = \mathbf{I}$. Expanding terms yields,

$$\hat{\varepsilon}_k = \|\mathbf{D}_k - \mathbf{H}_k\|_F^2 = tr\left[(\mathbf{D}_k - \mathbf{H}_k)^H(\mathbf{D}_k - \mathbf{H}_k)\right]$$
$$= \|\mathbf{D}_k\|_F^2 - 2\,\text{Re}\left\{tr\left[\mathbf{H}_k^H\mathbf{D}_k\right]\right\} - r$$
$$= \alpha - 2\,\text{Re}\left\{tr\left[\mathbf{H}_k^H\mathbf{D}_k\right]\right\}$$

where α is a constant. Thus, minimizing $\hat{\varepsilon}$ is equivalent to maximizing $\text{Re}\left\{tr\left[\mathbf{H}_k^H\mathbf{D}_k\right]\right\} \equiv \gamma$. Next, using a derivation similar to results from [3], [4], and [5], we start by taking the Singular Value Decomposition (SVD) of $\mathbf{D}_k$ to obtain

$$\mathbf{D}_k = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^H.$$

Here $\mathbf{U}_k$ is a p-by-p unitary matrix, $\mathbf{V}_k$ is a r-by-r unitary matrix and if the rank of $\mathbf{D}_k$ is ρ then $\mathbf{\Sigma}_k$ is a p-by-r diagonal matrix of singular values with $\sigma_0, \sigma_1, \ldots, \sigma_{\rho-1}$ on the diagonal, with all σ_i nonnegative. Then,

$$\gamma = \text{Re}\left\{tr\left[\mathbf{H}_k^H\mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^H\right]\right\}$$
$$= \text{Re}\left\{tr\left[\mathbf{\Sigma}_k\mathbf{V}_k^H\mathbf{H}_k^H\mathbf{U}_k\right]\right\}.$$

The matrix $\mathbf{P}_k = \mathbf{U}_k^H\mathbf{H}_k\mathbf{V}_k$ is unitary. Thus,

$$\gamma = \text{Re}\left\{tr\left[\mathbf{\Sigma}_k\mathbf{P}_k^H\right]\right\}$$
$$= \text{Re}\left\{\sum_{j=0}^{\rho-1}\sigma_j\left[\mathbf{P}_k\right]_{jj}^*\right\}$$
$$= \sum_{j=0}^{\rho-1}\sigma_j\,\text{Re}\left\{\left[\mathbf{P}_k\right]_{jj}^*\right\}$$
$$= \sum_{j=0}^{\rho-1}\sigma_j\,\text{Re}\left\{\left[\mathbf{P}_k\right]_{jj}\right\}.$$

Since $\mathbf{P}_k$ is unitary, its columns are orthonormal, which implies

$$\mathrm{Re}\{[\mathbf{P}_k]_{mn}\} \le 1 \quad \text{and}$$

$$\gamma \le \sum_{j=0}^{\rho-1} \sigma_j$$

with equality iff $[\mathbf{P}_k]_{mn} = \delta(m - n)$ for $0 \le m, n \le \rho - 1$. Thus the optimal p-by-r matrix $\mathbf{P}_k^{opt}$ is

$$\mathbf{P}_k^{opt} = \begin{bmatrix} \mathbf{I}_\rho & \mathbf{0}_{p\times(r-\rho)} \\ \mathbf{0}_{(p-\rho)\times\rho} & \mathbf{Q} \end{bmatrix}$$

where $\mathbf{Q}$ is an arbitrary $(p - \rho)$-by-$(r - \rho)$ unitary matrix such that $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}_{r-\rho}$. Lastly, since

$$\mathbf{P}_k = \mathbf{U}_k^H \mathbf{H}_k \mathbf{V}_k,$$

$$\mathbf{H}_k = \mathbf{U}_k \mathbf{P}_k \mathbf{V}_k^H \quad \text{so}$$

$$\mathbf{H}_k^{opt} = \mathbf{U}_k \mathbf{P}_k^{opt} \mathbf{V}_k^H.$$

The corresponding optimal value of $\hat{\varepsilon}$ is

$$\hat{\varepsilon}_k^{opt} = \alpha - 2\sum_{j=0}^{\rho-1} \sigma_j.$$

In the special case where $\rho = p = r$, then

$$\mathbf{H}_k^{opt} = \mathbf{U}_k \mathbf{V}_k^H,$$

$$\hat{\varepsilon}_k = \alpha - 2\sum_{j=0}^{r-1} \sigma_j.$$

These results suggest that it is possible to design a PUFB that satisfies the approximated MSE criterion well but with a transfer function $\mathbf{H}(\omega)$ that is guaranteed to be unitary only at discrete points on the unit circle. Nevertheless, the performance of such a filter bank may be more than adequate for a host of practical applications when the frequency grid is chosen large enough.

6. *DESIGN ALGORITHM*

In this section, an algorithm is presented to design a pointwise-in-frequency PUFB which satisfies an approximated MSE criterion and to derive the corresponding minimal time invariant realization. Assume $\mathbf{H}(z)$ is square for simplicity.

1. Given $\mathbf{D}(\omega)$, choose N, the number of points in the frequency grid.
2. For each k, $0 \le k \le N$-1, compute the SVD, $\mathbf{D}_k = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^H$ and set $\mathbf{H}_k = \mathbf{U}_k\mathbf{V}_k^H$.
3. Compute a finite Markov parameter sequence according to
$$\mathbf{H}_n \equiv \mathbf{H}(n) = \tfrac{1}{N}\sum_{k=0}^{N-1} \mathbf{H}_k e^{j2\pi kn/N}, \quad 0 \le n \le N-1.$$
4. Form the Hankel matrix,
$$\mathbf{\Gamma}_{NN} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{N-1} \\ \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{N-1} & \mathbf{H}_N & \cdots & \mathbf{H}_{2N-1} \end{bmatrix}.$$
Let n denote the rank of $\mathbf{\Gamma}_{NN}$.
5. Compute the linear system matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) corresponding to a minimal realization of $\mathbf{\Gamma}_{NN}$ with dimension n as described in Section 4, steps (*a*) through (*f*).
6. Compute the MIMO impulse response corresponding to ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) using
$$\mathbf{H}(n) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}, \quad n = 1, 2, \ldots N.$$

7. *DESIGN EXAMPLE*

For an example to illustrate our design procedure we consider Principal Component Filter Banks (PCFBs). PCFBs were originally described in [6]. It has been shown that PCFBs are optimal for maximizing coding gain and minimizing mean-squared error in the presence of quantization noise, and they are also optimal for any concave objective function of the subband variance vector [7]. Designing a PU approximation to a PCFB is a challenging problem because the response of an ideal PCFB channel filter may have several narrow bandpass regions.

The performance of the proposed design method is illustrated in Figures 1 through 4 and compared to an elegant algorithm presented by Tkacenko in [3] for approximating a PCFB using a PUFB that strictly satisfies the paraunitary condition everywhere. As the figures illustrate, the pointwise-in-frequency PUFB (red) very closely approximates the brick-wall response of the ideal filters (blue) on the selected frequency grid and is a better approximation to the ideal than the strictly PUFB (black).

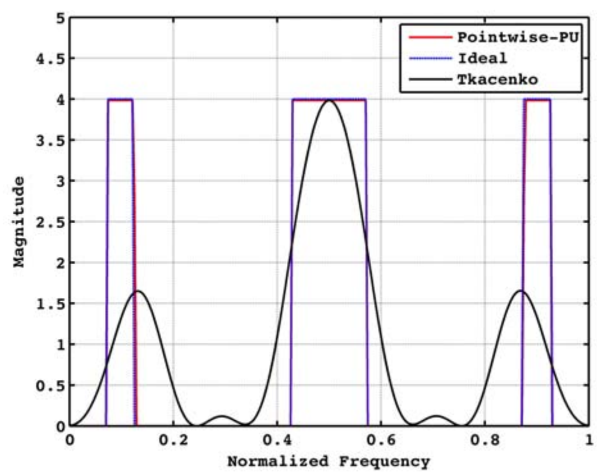

Fig. 1. Filter 1

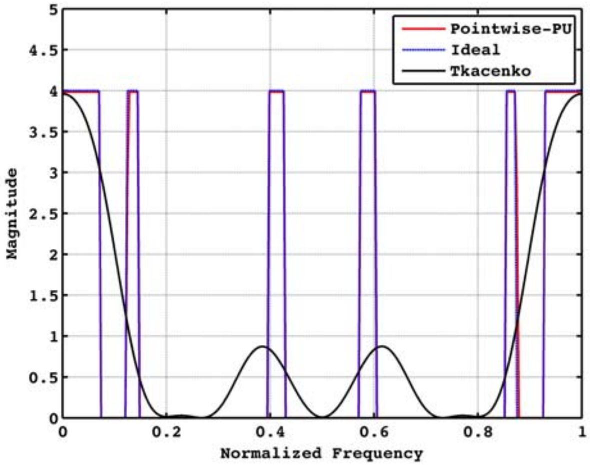

Fig. 2. Filter 2

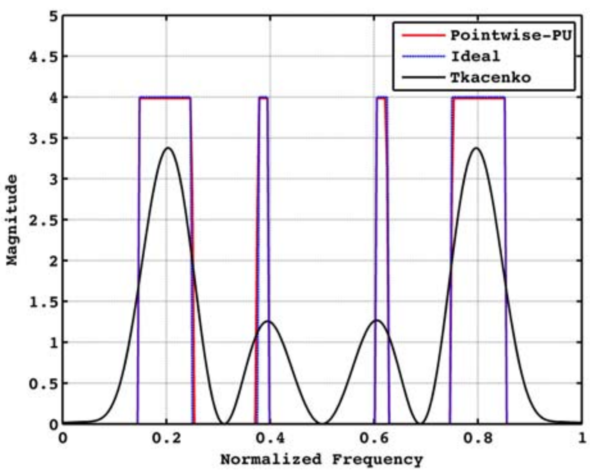

Fig. 3. Filter 3

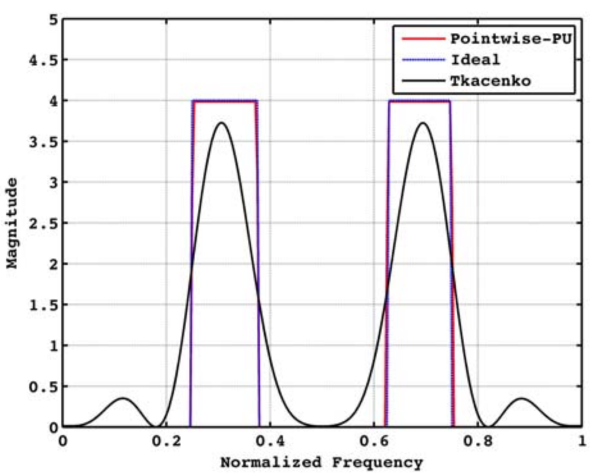

Fig. 4. Filter 4

8. *FREQUENCY ERRORS*

As Figures 1 through 4 illustrate, the proposed design technique yields an excellent filter bank approximation to the desired polyphase matrix at exactly the points of the selected frequency grid. Furthermore, the designed transfer matrix **H**(z) is unitary at those frequencies. In between the selected points on the frequency grid, the response of the designed filter bank will exhibit some errors in the passband and higher sidelobes in the stopband. The presence of these passband errors and higher sidelobes may or may not be important depending on the filter bank application. For example, if the filter bank is meant to partition the spectrum of noise-like interference at the input to a sidelobe canceller, then the passband errors and higher sidelobes may not degrade performance much. Figures 5 through 8 illustrate the response of each channel filter between the points on the frequency design grid and clearly illustrate the errors inherent in this filter bank design approach. Note that the error excursions in the pass band seem evenly centered about the desired passband response.

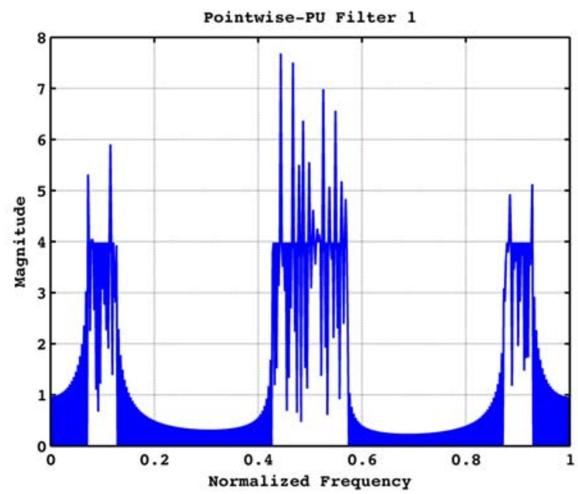

Fig. 5. Filter 1

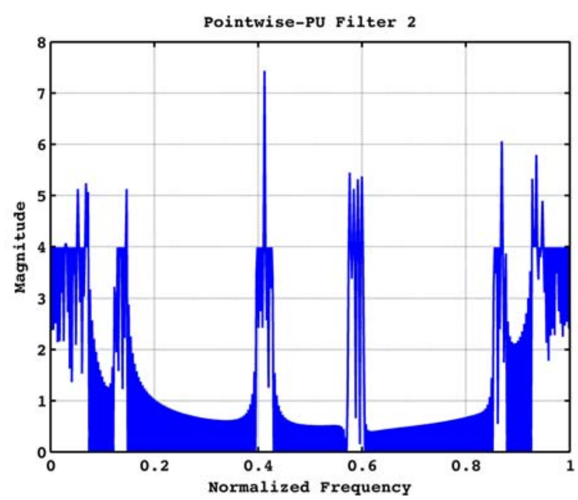

Fig. 6. Filter 2

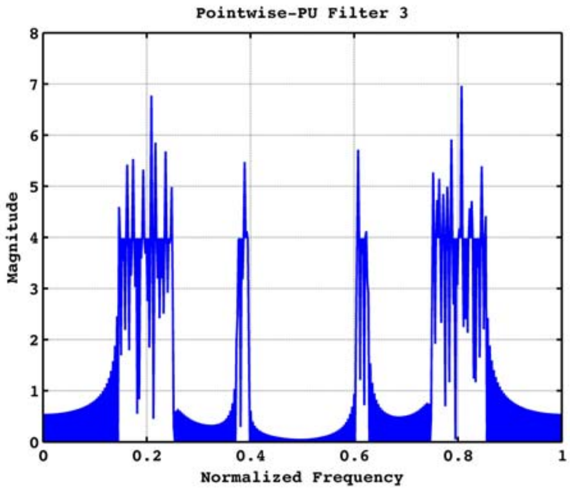

Fig. 7. Filter 3

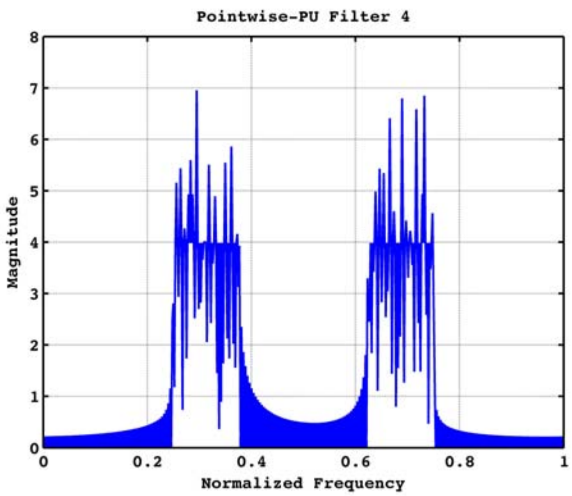

Fig. 8. Filter 4

9. *CONCLUSIONS*

In this paper, a new technique for designing approximations to PUFBs is presented whereby the transfer matrix is allowed to be unitary only on a discrete set of frequencies. These filter banks yield excellent approximations to the desired response at the selected frequencies and are much simpler to compute than strictly PUFBs, which require solving a nonlinear program using a parameterized representation of the PUFB.

10. *REFERENCES*

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